The 1-Lipschitz mapping between the unit spheres of two Hilbert spaces can be extended to a real linear isometry of the whole space

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Abstract Let $E$ and $F$ be Hilbert spaces with unit spheres $S_1(E)$ and $S_1(F)$. Suppose that $V_0: S_1(E) \to S_1(F)$ is a Lipschitz mapping with Lipschitz constant $k = 1$ such that $-V_0[S_1(E)] \subset V_0[S_1(E)]$. Then $V_0$ can be extended to a real linear isometric mapping $V$ from $E$ into $F$. In particular, every isometric mapping from $S_1(E)$ onto $S_1(F)$ can be extended to a real linear isometric mapping from $E$ onto $F$.

Keywords: isometric mapping, strictly convex, smooth point.

1 Preliminaries

Tingley proposed the following problem in ref. [1]: “Let $E$ and $F$ be real normed spaces with unit spheres $S_1(E)$ and $S_1(F)$. Suppose that $V_0: S_1(E) \to S_1(F)$ is a surjective isometry (i.e. $||V_0(x_1) - V_0(x_2)|| = ||x_1 - x_2||$ for all $x_1, x_2 \in S_1(E)$). Is $V_0$ necessarily the restriction to $S_1(E)$ of a linear or affine isometry on $E$?” (in the complex spaces, it is evident that the answer is negative. For example, we take $E = F = C$ (complex plane) and $V_0(x) = \bar{x}$).

In Tingley’s paper, he only obtained the affirmative answer on the assertion $V_0(-x) = -V_0(x)$ ($\forall x \in S_1(E)$) when the spaces are finite dimensional. In ref. [2], the above conclusion can also be obtained when the spaces are “strictly convex” Banach spaces. Recently, many papers have been published and the newest result has been obtained when $E$ is some Banach space and $F$ is $C(\Omega)$ (cf. ref. [3]).

Concerning isometry there is another problem: “What conditions are to be added to a (Dopp)-mapping $T$: $E \to F$ so that $T$ becomes an isometry?” (where $E$ and $F$ are real normed spaces and (Dopp) is the “distance one preserving property”; that is, for all $x_1, x_2 \in E$ with $||x_1 - x_2|| = 1$, $||T(x_1) - T(x_2)|| = 1$).

There are also many papers available concerning the above problem [4] and the newest result is as follows: “Let $E$ and $F$ be normed spaces with dim$E \geq 2$ such that one of them is strictly convex. Suppose that $T$: $E \to F$ is a homeomorphism satisfying the (Dopp). Then, $T$ is a linear isometry up to translation” [5].

Combining the above two problems, we can propose the following problem: “What conditions are to be added to a mapping $V_0$: $S_1(E) \to S_1(F)$ such that $V_0$ becomes a real linear isometry on $E$?” (where $E$ and $F$ are real or complex normed spaces).
In this paper, we shall demonstrate that if $V_0$ satisfies the Lipschitz condition with $k = 1$ and $-V_0[S_1(E)] \subset V_0[S_1(E)]$, then the answer of the above extension problem is affirmative when $E$ and $F$ are Hilbert spaces.

2 Main results

We shall first give two lemmas.

Lemma 2.1. Let $E$ and $F$ be normed spaces and $E$ be strictly convex, and $V_0$ a mapping from the unit sphere $S_1(E)$ into $S_1(F)$. If $-V_0[S_1(E)] \subset V_0[S_1(E)]$ and

$$||V_0(x_1) - V_0(x_2)|| \leq ||x_1 - x_2||, \quad \forall x_1, x_2 \in S_1(E),$$

then $V_0$ is one-to-one, and $V_0(-x) = -V_0(x)$ for all $x \in S_1(E)$.

Proof. For each $x \in S_1(E)$, let $y = V_0(x) \in S_1(F)$. Then, for every $x_1 \in V_0^{-1}(y)$ and $x_2 \in V_0^{-1}(-y)$, by the hypothesis of $V_0$ we have

$$2 \geq ||x_1|| + ||x_2|| \geq ||x_1 - x_2|| \geq ||V_0(x_1) - V_0(x_2)|| = ||y - (-y)|| = 2.$$

From the strictly convex condition of $E$, we get $x_1 = -x_2$, and hence $V_0$ is one-to-one and $x = x_1 = -x_2$. Moreover, from $-x = x_2$ we know

$$V_0(-x) = V_0(x_2) = V_0[V_0^{-1}(-y)] = -y = -V_0(x).$$

That is, we obtain

$$V_0(-x) = -V_0(x), \quad \forall x \in S_1(E).$$

Note. The condition $-V_0[S_1(E)] \subset V_0[S_1(E)]$ in the above lemma is not removable, since a counterexample can be given as follows: Take $V_0(x) \equiv y_0$ ($y_0$ is some element of $S_1(F)$) for all $x \in S_1(E)$.

Lemma 2.2. In Lemma 2.1, if both $E$ and $F$ are inner-product spaces, we have

$$||V_0(x_1) - \lambda V_0(x_2)|| = ||x_1 - \lambda x_2||, \quad \forall \lambda \in R, \ \forall x_1, x_2 \in S_1(E).$$

Proof. Firstly, we prove that $V_0$ is isometric on $S_1(E)$. In fact, $x_1^0, x_2^0 \in S_1(E)$ such that $||V_0(x_1^0) - V_0(x_2^0)|| \neq ||x_1^0 - x_2^0||$, and then

$$||V_0(x_1^0) - V_0(x_2^0)|| < ||x_1^0 - x_2^0||. \quad (2.1)$$

By the parallelogram law of norm in the inner-product space we have

$$4 = 2(||V_0(x_1^0)||^2 + ||V_0(x_2^0)||^2)
\leq ||V_0(x_1^0) + V_0(x_2^0)||^2 + ||V_0(x_1^0) - V_0(x_2^0)||^2
< ||V_0(x_1^0) + V_0(x_2^0)||^2 + ||x_1^0 - x_2^0||^2. \quad (2.2)$$

From Lemma 2.1, we have

$$||V_0(x_1^0) + V_0(x_2^0)|| = ||V_0(x_1^0) - V_0(-x_2^0)|| \leq ||x_1^0 - (-x_2^0)|| = ||x_1^0 + x_2^0||. \quad (2.3)$$

From (2.2) and (2.3) we derive

$$4 < ||x_1^0 + x_2^0||^2 + ||x_1^0 - x_2^0||^2 = 2(||x_1^0||^2 + ||x_2^0||^2) = 4,$$

which contradicts (2.1). Thus we obtain

$$||V_0(x_1) - V_0(x_2)|| = ||x_1 - x_2||, \quad \forall x_1, x_2 \in S_1(E). \quad (2.4)$$
Secondly, from (2.4) we have
\[(V_0(x_1) - V_0(x_2), V_0(x_1) - V_0(x_2)) = (x_1 - x_2, x_1 - x_2), \quad \forall x_1, x_2 \in S_1(E),\]
i.e.
\[2 - (V_0(x_1), V_0(x_2)) - (V_0(x_2), V_0(x_1)) = 2 - (x_1, x_2) - (x_2, x_1).\]
Then
\[(V_0(x_1), V_0(x_2)) + (V_0(x_2), V_0(x_1)) = (x_1, x_2) + (x_2, x_1), \quad \forall x_1, x_2 \in S_1(E). \tag{2.5}\]
By (2.5), for each \(\lambda \in R\) we have
\[
(V_0(x_1), V_0(x_2)) - \lambda(V_0(x_1), V_0(x_2)) - \lambda(V_0(x_2), V_0(x_1)) + \lambda^2(V_0(x_2), V_0(x_2))
= 1 + \lambda^2 - \lambda(V_0(x_1), V_0(x_2)) - \lambda(V_0(x_2), V_0(x_1))
= 1 + \lambda^2 - \lambda(x_1, x_2) - \lambda(x_2, x_1)
= (x_1, x_2) - \lambda(x_1, x_2) - \lambda(x_2, x_1) + \lambda^2(x_2, x_2).
\]
Hence, we obtain
\[\|V_0(x_1) - \lambda V_0(x_2)\| = \|x_1 - \lambda x_2\|, \quad \forall \lambda \in R, \forall x_1, x_2 \in S_1(E).\]
Next we shall give the first theorem.

**Theorem 2.1.** Let \(E\) and \(F\) be inner-product spaces, and \(V_0\) a mapping from the unit sphere \(S_1(E)\) into \(S_1(F)\). If \(-V_0[S_1(E)] \subset V_0[S_1(E)]\) and
\[\|V_0(x_1) - V_0(x_2)\| \leq \|x_1 - x_2\|, \quad \forall x_1, x_2 \in S_1(E),\]
then \(V_0\) can be extended to be a real homogeneous isometric mapping of \(E\) into \(F\).

**Proof.** Let
\[V(z) = \begin{cases} \|z\| V_0\left(\frac{z}{\|z\|}\right), & \text{if } z \neq \theta, \\ \theta, & \text{if } z = \theta. \end{cases} \tag{2.6}\]
Then, by Lemma 2.1 for every \(0 \neq \lambda \in R\), we have
\[V(\lambda z) = \|\lambda z\| V_0\left(\frac{\lambda z}{\|\lambda z\|}\right) = \|\lambda z\| \left(\frac{\lambda}{\|\lambda z\|}\right) V_0\left(\frac{z}{\|z\|}\right) = \lambda V(z), \quad \forall \theta \neq z \in E,\]
i.e. \(V\) is real homogeneous.

By Lemma 2.2, if \(z_1 \neq \theta, z_2 \neq \theta\), we get
\[
\|V(z_1) - V(z_2)\| = \left\|\|z_1\| V_0\left(\frac{z_1}{\|z_1\|}\right) - \|z_2\| V_0\left(\frac{z_2}{\|z_2\|}\right)\right\|
= \|z_1\| \left\|V_0\left(\frac{z_1}{\|z_1\|}\right) - \frac{\|z_1\|}{\|z_1\|} V_0\left(\frac{z_2}{\|z_2\|}\right)\right\|
= \|z_1\| \left\|z_1 - \frac{\|z_2\|}{\|z_1\|} z_2\right\| = \|z_1 - z_2\|, \quad \forall z_1, z_2 \in E.
\]
Thus, \(V\) is isometric on \(E\).

For the linearity of the above \(V\) we need to introduce the following lemma (Lemma 9.4.6 in ref. [6]).

**Lemma 2.3.** Let \(V\) be an “isometry” of a real Banach space \(E\) into a real Banach space \(F\) such that \(V(\theta) = \theta\). Let \(x_0\) be a "smooth point" of the sphere \(S_{\|x_0\|}\) and \(g \in F^*\) be a continuous linear functional of norm one such that, for all real \(\lambda\),
\[g[V(\lambda x_0)] = \lambda \|x_0\|.\]
Then

\[ g[V(x)] = f_{x_0}(x), \quad \forall x \in E, \]

where \( f_{x_0} \in E^* \) is the support functional at \( x_0 \) in the sphere \( S_{||x_0||} \).

**Theorem 2.2.** Let \( E \) and \( F \) be Hilbert spaces, and \( V_0 \) a mapping from the unit sphere \( S_1(E) \) into \( S_1(F) \). If \( -V_0[S_1(E)] \subset V_0[S_1(E)] \) and

\[ ||V_0(x_1) - V_0(x_2)|| \leq ||x_1 - x_2||, \quad \forall x_1, x_2 \in S_1(E). \]

Then \( V_0 \) can be extended to be a real linear isometric mapping from \( E \) into \( F \).

**Proof.** Let \( V(z) \) be the same as in ref. [6] in the proof of Theorem 2.1. Then we know that \( V(z) \) is a real homogeneous isometric mapping from \( E \) to \( F \).

Next we shall prove the additivity of \( V(z) \) on \( E \).

First, we notice that every point \( x \in S_1(E) \) is a smooth point of \( S_1(E) \) since \( E \) is a Hilbert space. By the Riesz representation theorem, for each \( V(x) \in F \), we can assume \( g = V(x) \in F^* \), and we have

\[ g[V(\lambda z)] = (V(\lambda z), V(x)) = (\lambda V(x), V(x)) = \lambda ||V(x)||^2 = \lambda = \lambda ||V(x)||, \quad \forall \lambda \in R. \]

Hence, by Lemma 2.3 we obtain

\[ g[V(z)] = f_x(z), \quad \forall z \in E; \]

that is, for every \( x \in S_1(E) \),

\[ (V(z), V(x)) = f_x(z), \quad \forall z \in E, \quad (2.7) \]

where \( f_x \in E^* \) is the support functional at \( x \) in the unit sphere \( S_1(E) \).

By (2.7), it is easy to verify that for \( z_1, z_2 \in E \) we have

\( (V(z_1 + z_2), V(x)) = (V(z_1) + V(z_2), V(x)), \quad \forall x \in S_1(E), \)

\( (V(z_1 + z_2), \alpha V(x_1) + \beta V(x_2)) = (V(z_1) + V(z_2), \alpha V(x_1) + \beta V(x_2)), \quad \forall x_1, x_2 \in S_1(E), \quad \forall \alpha, \beta \in R \)

and

\[ (V(z_1 + z_2), y) = (V(z_1) + V(z_2), y), \quad \forall y \in \text{Span}\{V(x)|x \in S_1(E)\}. \quad (2.8) \]

Let

\[ F_0 = \text{Span}\{V(x)|x \in S_1(E)\}. \]

Then, \( F_0 \) is also a Hilbert space and \( \text{Span}\{V(z)|z \in E\} \subset F_0 \), so \( F_0^* = F_0 \). By (2.8) and the Hahn-Banach theorem we obtain

\[ V(z_1 + z_2) = V(z_1) + V(z_2), \quad \forall z_1, z_2 \in E. \]

Thus, \( V \) is a real linear isometric mapping from \( E \) into \( F \).

**Corollary 1.** Let \( E \) and \( F \) be Hilbert spaces, and \( V_0 \) a 1-Lipschitz mapping from \( E \) “onto” \( F \). Then \( V_0 \) can be extended to a real linear isometric mapping from \( E \) onto \( F \).

When we notice that if \( F \) is strictly convex and \( V_0 \) is isometric, then \(-V_0[S_1(E)] \subset V_0[S_1(E)]\).

Thus we have

**Corollary 2.** Let \( E \) and \( F \) be Hilbert spaces, and \( V_0 \) an isometric mapping from \( E \) “into” \( F \). Then, \( V_0 \) can be extended to a real linear isometric mapping from \( E \) into \( F \).
Remark. It is clear that the above Theorem 2.1 implies Theorem 2.2 by Baker’s theorem\cite{7}. However, for Hilbert spaces, the above independent proof seems to be better.

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References